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A MORSE FUNCTION ON GRASSMANN MANIFOLDS

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Studying the critical sections of a convex body Wen Tsun Wu has obtained in [2] a Morse function on a Grassmann manifold. In the sequel it will be shown that another function may be obtained by composing the embedding of this manifold into a projective space with the well known Morse function of the projective space; our work is valid only for the real and complex fields.

1. The homology of the Grassmann manifold $G_{p,q}$ of all the p-planes of codimension q which pass through a fixed point 0 in an affine space A^n of dimension $n = p + q$ was determined in 1934 by Ch. Ehresmann who gave a cell subdivision of $G_{p,q}$. The number of cells in his subdivision is the number $N = \binom{n}{p}$ of combinations of p elements of the set $\{1, \cdots n\}$; such a combination $\sigma = (\sigma_1, \cdots, \sigma_p)$ where $1 \leq \sigma_1 < \cdots < \sigma_p \leq n$ is called a Schubert symbol. In the cell-subdivision of $G_{p,q}$, with each symbol σ one associates a cell of dimension

$$d(\sigma) = (\sigma_1 - 1) + \cdots + (\sigma_p - p) .$$

Let us consider the lexicographical order in the set $S(p, q)$ of all the Schubert symbols which correspond to the integers p and q; this means that $\sigma = (\sigma_1, \cdots, \sigma_p) < \sigma' = (\sigma'_1, \cdots, \sigma'_p)$ if and only if for the least integer $i \leq p$ for which $\sigma_i \neq \sigma'_i$ the inequality $\sigma_i < \sigma'_i$ holds. We say that two symbols $\sigma = (\sigma_1, \cdots, \sigma_p)$ and $\sigma' = (\sigma'_1, \cdots, \sigma'_p)$ are *neighboring* if the sets $\{\sigma_1, \cdots, \sigma_p\}$ and $\{\sigma'_1, \cdots, \sigma'_p\}$ have exactly $p - 1$ elements in common, or equivalently, if they differ only in what a single element is concerned. With these conventions we observe that the number $d(\sigma)$ equals the number of those Schubert-symbols which are less than and neighboring to σ. Indeed, in order to obtain a new symbol less than and neighboring to σ, the change of σ_i in σ may be made in $\sigma_i - i$ ways by replacing σ_i with a positive integer less than σ_i and different from $\sigma_1, \cdots, \sigma_{i-1}$.

2. In the projective space P^{N-1} of dimension $N - 1$ we consider homogeneous coordinates y_σ having as indices Schubert symbols $\sigma \in S(p, q)$ instead of positive integers running from 1 to N.

It is known, for example from [1], that the function

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$$f = \sum_{\sigma \in S(p,q)} c_\sigma |y_\sigma|^2 , \tag{1}$$

where c_σ are constants, and satisfy the inequalities $c_\sigma < c_{\sigma'}$ when $\sigma < \sigma'$ defines a Morse function on P^{N-1} when the coordinates y_σ satisfy the equation

$$\sum_{\sigma \in S(p,q)} |y_\sigma|^2 = 1 . \tag{2}$$

The critical points of this function f correspond to the coordinate axes in the numerical N-dimensional space of the variables y_σ, and therefore may be denoted by A_σ, $\sigma \in S(p, q)$. The index of the point A_σ corresponding to the y_σ-axis is equal to and twice the number of constants $c_{\sigma'}$, which are less than c_σ, in the real and complex cases respectively. In other words, this index equals $n_\sigma - 1$ in the real case and $2(n_\sigma - 1)$ in the complex case, where n_σ is the number associated with σ in the ordering of $S(p, q)$.

3. In the affine space A^n of dimension n denote by e_a, $a = 1, \cdots, n$, the basis vectors of the system of cartesian coordinates having the origin at 0. Consider p linearly independent vectors v_α, $\alpha = 1, \cdots, p$, with components with respect to the basis $\{e_a\}$ denoted by $v_\alpha^a \left(v_\alpha = \sum_{a=1}^{n} v_\alpha^a e_a\right)$ and form the determinants

$$v^\sigma = \det\|v_\alpha^{\sigma_\alpha}\| , \qquad \alpha = 1, \cdots, p, \quad \sigma \in S(p, q) , \tag{3}$$

which realize a system of Plücker coordinates for the p-plane spanned by the p vectors v_α. The Plückerian embedding π of $G_{p,q}$ in P^{N-1} is given by the equations

$$y_\sigma = v^\sigma . \tag{4}$$

Observe that when $v_\alpha = e_{\sigma_\alpha}$ the corresponding p-plane has the only non-zero component $y_\sigma = 1$ and thus the points A_σ, which are critical points for the function f, belong to the image $\pi(G_{p,q})$.

Theorem. *The function* $f \circ \pi \colon G_{p,q} \to R$ *is a Morse function having* $N = \binom{n}{p}$ *nondegenerate critical points, which are* $\pi^{-1}(A_\sigma)$, *and the index of each such point is* $d(\sigma)$ *in the real case and* $2d(\sigma)$ *in the complex case.*

4. The points $\pi^{-1}(A_\sigma)$ are critical for the function $f \circ \pi$ since their images A_σ are so for the function f. In order to show that the critical points $\pi^{-1}(A_\sigma)$ are nondegenerate and their index is $d(\sigma)$, we introduce a system of local coordinates on $G_{p,q}$ in the neighborhood U_σ of the p-plane $\pi^{-1}(A_\sigma)$ whose points are the p-planes having a nondegenerate projection on $\pi^{-1}(A_\sigma)$. Clearly, if $e_{\bar\sigma_i}$, $i = p + 1, \cdots, p + q$, $1 \le \bar\sigma_i \le n$, $\bar\sigma_i \ne \sigma_\alpha$, are the vectors of the already chosen basis in A^n, which are not in $\pi^{-1}(A_\sigma)$, then the pq local coordinates x_α^i of a point x belonging to U_σ are determined by the formulas

$$v_\alpha = e_{\sigma_\alpha} + \sum_{i=p+1}^{n} x_\alpha^i e_{\bar{\sigma}_i}, \qquad \alpha = 1, \cdots, p ,$$

v_α being the generating vectors of x. Observe now that $v^\sigma = 1$ and that the only determinats v^ρ, $\rho \in S(p, q)$, which are linear functions of the coordinates x_α^i, are those corresponding to the symbols ρ which are neighboring to σ. The other determinants v^ρ are homogeneous polynominals in x_α^i of degree greater than one. In order for the embedding π: $G_{p,q} \to P^{N-1}$ to satisfy the condition (2) we use the following formulas:

$$(5) \qquad y_\rho = \frac{v^\rho}{\left(\sum_{\tau \in S(p,q)} |v^\tau|^2\right)^{1/2}} .$$

Thus the function $F = f \circ \pi$ becomes

$$(6) \qquad F = \frac{\sum_{\rho \in S(p,q)} c_\rho |v^\rho|^2}{\sum_{\tau \in S(p,q)} |v^\tau|^2} ,$$

and at the origin of the system of coordinates x_α^i the value of this function F is c_σ. This point is a critical one and the quadratic form F_σ, which approximates the function $F - c_\sigma$ in the neighborhood of the origin, is

$$F_\sigma = \sum_{\sigma'} (c_{\sigma'} - c_\sigma) |v^{\sigma'}|^2 ,$$

where σ' is neighboring to σ. $|v^{\sigma'}|^2$ is the square of one of the coordinates x_α^i in the real case, and is its modulus $(\mathrm{Re}\, x_\alpha^i)^2 + (\mathrm{Im}\, x_\alpha^i)^2$ in the complex case. Hence the last part of the theorem follows from the choice of the constants c_ρ.

5. It remains to be proved that the function F has no other critical points different from $\pi^{-1}(A_\sigma)$. In order to do this suppose that v is a critical point for F, and that σ is the least Schubert symbol having the property that the p-plane v belongs to U_σ. Thus the matrix of the components of a system of p vectors v_α which span the p-plane v in U_σ is of the form

$$(7) \qquad \begin{pmatrix} 0 \cdots 0 & 1\, v_1^{\sigma_1+1} \cdots 0 & v_1^{\sigma_2+1} \cdots 0 & \cdots v_1^n \\ 0 \cdots\cdot & \cdots\ \cdot\, 0\, 1 & v_2^{\sigma_2+1} \cdots 0 & \cdots v_2^n \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdots\ 0\, 1\, v_p^{\sigma_p+1} & \cdots v_p^n \end{pmatrix} .$$

Clearly $v_\alpha^a = 0$, $a < \sigma_\alpha$, and $v_\alpha^{\sigma_\beta} = \delta_\alpha^\beta$ where δ_α^β is the Kronecker symbol. Now we consider the curve w: $(-\varepsilon, \varepsilon) \to G_{p,q}$ obtained by keeping the vectors $v_2, \cdots, v_p$ constant and varying only v_1 in accord with the formulas

$$(8) \qquad w_1^i = v_1^i + t v_1^i,\ i \neq \sigma_1;\ w_1^{\sigma_1} = v_1^{\sigma_1} = 1,\ w_\alpha = v_\alpha,\ \alpha = 2, \cdots, p.$$

Thus, when v_1^i, $i \neq \sigma_1$, are not all zero, $\left(\dfrac{dF(w(t))}{dt}\right)_{t=0} \neq 0$ which contradicts the hypothesis that v is a critical point. Indeed, from (6) we obtain

$$\left(\frac{dF(w(t))}{dt}\right)_{t=0} = 2\,\mathrm{Re}\,\frac{\left(\sum_{\rho} c_\rho v^\rho \bar{w}_0^{\rho'}\right)\left(\sum_{\tau} |v^\tau|^2\right) - \left(\sum_{\tau} v^\tau \bar{w}_0^{\tau'}\right)\left(\sum_{\rho} c_\rho |v^\rho|^2\right)}{\left(\sum_{\tau} |v^\tau|^2\right)^2}, \tag{9}$$

where $w_0^{\rho'} = \left(\dfrac{dw^\rho}{dt}\right)_{t=0}$. From (7) and (8) we observe that if $w_0^{\rho'} \neq 0$, then the symbol $\rho = (\rho_1, \cdots, \rho_p)$ must have $\rho_1 > \sigma_1$ and in this case $w_0^{\rho'} = v^\rho$. We write such a symbol in the form $\rho = \rho_1\bar{\rho}$ where $\bar{\rho} \in S(p-1, q+1)$ is the Schubert symbol $\rho = (\rho_2, \cdots, \rho_p)$. With this convention the numerator on the right-hand side of (9) then becomes

$$\begin{aligned}
\mathfrak{N}_1 &= \left(\sum_{\rho_1>\sigma_1,\bar{\rho}} c_{\rho_1\bar{\rho}} |v^{\rho_1\bar{\rho}}|^2\right)\left(\sum_{\tau_1>\sigma_1,\bar{\tau}} |v^{\tau_1\bar{\tau}}|^2 + \sum_{\bar{\tau}} |v^{\sigma_1\bar{\tau}}|^2\right) \\
&\quad - \left(\sum_{\tau_1>\sigma_1,\bar{\tau}} |v^{\tau_1\bar{\tau}}|^2\right)\left(\sum_{\rho_1>\sigma_1,\bar{\rho}} c_{\rho_1\bar{\rho}} |v^{\rho_1\bar{\rho}}|^2 + \sum_{\bar{\tau}} c_{\sigma_1\bar{\tau}} |v^{\sigma_1\bar{\tau}}|^2\right) \\
&= \sum_{\rho_1>\sigma_1,\bar{\rho},\bar{\tau}} (c_{\rho_1\bar{\rho}} - c_{\sigma_1\bar{\tau}}) |v^{\rho_1\bar{\rho}}|^2 |v^{\sigma_1\bar{\tau}}|^2 .
\end{aligned}$$

But $c_{\rho_1\bar{\rho}} > c_{\sigma_1\bar{\tau}}$ since $\rho_1 > \sigma_1$, and as among the components $v^{\sigma_1\bar{\tau}}$ there is at least one different from zero (the component $v^\sigma = 1$) we infer that $\mathfrak{N}_1 = 0$ only if all the determinants $v^{\rho_1\bar{\rho}}$ vanish. Among these determinants v^ρ we find those, for which $p-1$ indices in the symbol ρ coincide with $\sigma_2, \cdots, \sigma_p$ and are equal to $\pm v_1^i$, $i > \sigma_1$. Thus, if v is a critical point for F, then its coordinates v_1^i, $i > \sigma_1$, must vanish. The same method may be used to show that all the components v_α^i, $i > \sigma_\alpha$, vanish for a critical point v. Indeed suppose that for a critical point v we have

$$v_\alpha^i = 0,\ i > \sigma_\alpha,\ \alpha = 1, \cdots, k-1 < p\ , \tag{10}$$

and consider the curve $w\colon (-\varepsilon, \varepsilon) \to G_{p,q}$ defined by

$$w_k^i = v_k^i + t v_k^i,\ i \neq \sigma_k, \quad w_k^{\sigma k} = v_k^{\sigma k}, \quad w_\beta = v_\beta,\ \beta \neq k\ . \tag{11}$$

From (10), (11) and (7) we infer that the components v^ρ where ρ is not of the form $\rho = (\sigma_1, \cdots, \sigma_{k-1}, \rho_k, \cdots, \rho_p)$ are zero, and that the derivatives $w_0^{\rho'} = \left(\dfrac{dw^\rho}{dt}\right)_{t=0}$ where $\rho_k = \sigma_k$ are also zero. Thus $\mathfrak{N}_1$, now denoted by $\mathfrak{N}_k$, becomes

$$\mathfrak{N}_k = \Big(\sum_{\rho_k>\sigma_k,\bar\rho} c_{\sigma_1\cdots\sigma_{k-1}\rho_k\bar\rho}|v^{\sigma_1\cdots\sigma_{k-1}\rho_k\bar\rho}|^2\Big)\Big(\sum_{\tau_k>\sigma_k,\bar\tau}|v^{\sigma_1\cdots\sigma_{k-1}\tau_k\bar\tau}|^2 + \sum_{\bar\pi}|v^{\sigma_1\cdots\sigma_k\bar\pi}|^2\Big)$$
$$- \Big(\sum_{\tau_k>\sigma_k,\bar\tau}|v^{\sigma_1\cdots\sigma_{k-1}\tau_k\bar\tau}|^2\Big)\Big(\sum_{\rho_k>\sigma_k,\bar\rho} c_{\sigma_1\cdots\sigma_{k-1}\rho_k\bar\rho}|v^{\sigma_1\cdots\sigma_{k-1}\rho_k\bar\rho}|^2$$
$$+ \sum_{\bar\pi} c_{\sigma_1\cdots\sigma_k\bar\pi}|v^{\sigma_1\cdots\sigma_k\bar\pi}|^2\Big)$$
$$= \sum_{\rho_k>\sigma_k,\bar\rho,\bar\pi}(c_{\sigma_1\cdots\sigma_{k-1}\rho_k\bar\rho} - c_{\sigma_1\cdots\sigma_k\bar\pi})|v^{\sigma_1\cdots\sigma_{k-1}\rho_k\bar\rho}|^2|v^{\sigma_1\cdots\sigma_k\bar\pi}|^2 ,$$

where $\bar\rho$ denotes $\rho_{k+1}\cdots\rho_p$ for abbreviation. As above $\mathfrak{N}_k = 0$ implies $v_k^i = 0$, $i > \sigma_k$. Hence the only critical points F are the points $\pi^{-1}(A_\sigma)$.

Bibliography

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